CONTRACTING AN ELEMENT FROM A COCIRCUIT.

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Abstract. We consider the situation that M and N are 3-connected matroids such that $|E(N)| \geq 4$ and C^* is a cocircuit of M with the property that M/x_0 has an N-minor for some $x_0 \in C^*$. We show that either there is an element $x \in C^*$ such that $\mathrm{si}(M/x)$ or $\mathrm{co}(\mathrm{si}(M/x))$ is 3-connected with an N-minor, or there is a four-element fan of M that contains two elements of C^* and an element x such that $\mathrm{si}(M/x)$ is 3-connected with an N-minor.

1. Introduction

There are a number of tools in matroid theory that tell us when we can remove an element or elements from a matroid, while maintaining both the presence of a minor and a certain type of connectivity. Some recent results are of this type, but have the additional restriction that the element(s) must have a certain relation to a given substructure in the matroid. For example, Oxley, Semple, and Whittle [9], consider a given basis of a matroid and consider either contracting elements that are in the basis, or deleting elements that are not in the basis. Hall [3] has investigated when it is possible to contract an element from a given hyperplane in a 3-connected matroid and remain 3-connected (up to parallel pairs).

We make a contribution to this collection of tools by investigating the circumstances under which we can contract an element from a cocircuit while maintaining both the presence of a minor and 3-connectivity (up to parallel pairs), and the structures which prevent us from doing so. Our result has been employed by Geelen, Gerards, and Whittle [2] in their characterization of when three elements in a matroid lie in a common circuit.

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Theorem 1.1. *Suppose that M and N are 3-connected matroids such that $|E(N)| \geq 4$ and C^* is a cocircuit of M with the property that M/x_0 has an N-minor for some $x_0 \in C^*$. Then either:*

(i) there is an element $x \in C^*$ such that $\mathrm{si}(M/x)$ is 3-connected and has an N-minor;
(ii) there is an element $x \in C^*$ such that $\mathrm{co}(\mathrm{si}(M/x))$ is 3-connected and has an N-minor; or,
(iii) there is a sequence of elements (x_1, x_2, x_3, x_4) from $E(M)$ such that $\{x_1, x_2, x_3\}$ is a circuit, $\{x_2, x_3, x_4\}$ is a cocircuit, $x_1, x_3 \in C^*$, and $\mathrm{si}(M/x_2)$ is 3-connected with an N-minor.

The next example shows that statement (ii) of Theorem 1.1 is necessary.

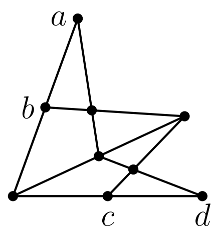

FIGURE 1. The graphic matroid $M(K_5\backslash e)$.

Consider the rank-4 matroid M whose geometric representation is shown in Figure 1. Note that $M \cong M(K_5\backslash e)$. The set $C = \{a, b, c, d\}$ is a circuit of M, and hence a cocircuit of M^*. Moreover M^*/x has a minor isomorphic to $M(K_4)$ for any element $x \in C$. However $\mathrm{co}(M\backslash x)$ is not 3-connected, as it contains a parallel pair, so $\mathrm{si}(M^*/x)$ is not 3-connected. On the other hand $\mathrm{co}(\mathrm{si}(M^*/x))$ is 3-connected, and has a minor isomorphic to $M(K_4)$.

More generally we suppose that r is an integer greater than two. Consider a basis $A = \{a_1, \ldots, a_r\}$ in the projective space $\mathrm{PG}(r-1, \mathbb{R})$. Let l be a line of $\mathrm{PG}(r-1, \mathbb{R})$ that is freely placed relative to A, and for all $i \in \{1, \ldots, r\}$ let b_i be the point that is in both l and the hyperplane of $\mathrm{PG}(r-1, \mathbb{R})$ spanned by $A - a_i$. Let $B = \{b_1, \ldots, b_r\}$. We will use Θ_r to denote the restriction of $\mathrm{PG}(r-1, \mathbb{R})$ to $A \cup B$.

Suppose that Θ_r' is an isomorphic copy of Θ_r with $\{a_1', \ldots, a_r'\} \cup B$ as its ground set. Assume also that the isomorphism from Θ_r to Θ_r' acts as the identity on B and takes a_i to a_i' for all $i \in \{1, \ldots, r\}$. Let M be the generalized parallel connection of Θ_r and Θ_r'. That is, M is a matroid on the ground set $A \cup A' \cup B$ and the flats of M are exactly the sets F such that $F \cap (A \cup B)$ is a flat of Θ_r and $F \cap (A' \cup B)$ is a

flat of Θ'_r. Note that if $r = 3$ then M is isomorphic to $M(K_5\backslash e)$, the matroid illustrated in Figure 1.

It is easy to see that Θ_r is self-dual and that $C = (A - a_1) \cup (A' - a'_1)$ is a circuit of M, and hence a cocircuit of M^*. Moreover M^*/x has an isomorphic copy of Θ_r as a minor for every element $x \in C$. We note that every three-element subset of A is a circuit of M^*. Thus $A - x$ is a parallel class of M^*/x for every $x \in C \cap A$. However the simplification of M^*/x contains a unique series pair, and is therefore not 3-connected. On the other hand $\mathrm{co}(\mathrm{si}(M^*/x))$ is 3-connected, and has a minor isomorphic to Θ_r.

The structure described in the last example has been discovered before. The matroid Θ_r is a fundamental object in the generalized Δ-Y operation of Oxley, Semple, and Vertigan [7]. Furthermore this construction is an example of a 'crocodile', as described by Hall, Oxley, and Semple [4].

To see that statement (iii) of Theorem 1.1 is necessary consider the graph G shown in Figure 2. Let C^* be the cocircuit of $M = M(G)$ comprising the edges incident with the vertex a. It is easy to see that if x is any edge between a and a vertex in $\{b, c, d, e, f\}$ then M/x has a minor isomorphic to $M(K_6)$, and that these are the only edges in C^* with this property. But in this case neither $\mathrm{si}(M/x)$ nor $\mathrm{co}(\mathrm{si}(M/x))$ is 3-connected. On the other hand, if we let x_1 be the edge ad, x_2 be cd, x_3 be ac, and x_4 be bc, then (x_1, x_2, x_3, x_4) is a sequence of the type described in statement (iii) of Theorem 1.1.

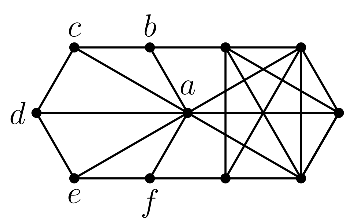

FIGURE 2. The graph G.

Our main result shows that there are essentially only two structures that prevent us from finding an element $x \in C^*$ such that $\mathrm{si}(M/x)$ is 3-connected with an N-minor. These structures are named 'segment-cosegment pairs' and 'four-element fans'. The dual of the matroid in Figure 1 contains a segment-cosegment pair, and the graph in Figure 2 contains a four-element fan. Before describing our result in detail we fix some terminology. Suppose that M is a matroid. Recall that a *triangle* of M is a three-element circuit, and a *triad* is a three-element

cocircuit. A *four-element fan* of M is a sequence (x_1, x_2, x_3, x_4) of distinct elements from $E(M)$ such that $\{x_1, x_2, x_3\}$ is a triangle and $\{x_2, x_3, x_4\}$ is a triad. A *segment* of M is a set L such that $|L| \geq 3$ and every three-element subset of M is a triangle, and a *cosegment* of M is a segment of M^*. We say that (L, L^*) is a *segment-cosegment pair* if $L = \{x_1, \ldots, x_t\}$ is a segment of M, and $L^* = \{y_1, \ldots, y_t\}$ is a set such that $L \cap L^* = \emptyset$ and for every $x_i \in L$ the set $(\mathrm{cl}(L) - x_i) \cup y_i$ is a cocircuit. Segment-cosegment pairs will be considered in detail in Section 3. A *spore* is a pair (P, s) such that P is a rank-one flat, and $P \cup s$ is a cocircuit. A matroid M is 3-*connected up to a unique spore* if M contains a single spore (P, s), and whenever (X, Y) is a k-separation of M for some $k < 3$ then either $X \subseteq P \cup s$ or $Y \subseteq P \cup s$. Theorem 1.1 follows from the next result. It gives a more detailed analysis of the structures we encounter.

Theorem 1.2. *Suppose that M and N are 3-connected matroids such that $|E(N)| \geq 4$ and C^* is a cocircuit of M with the property that M/x_0 has an N-minor for some $x_0 \in C^*$. Then either:*

(i) there is an element $x \in C^*$ such that $\mathrm{si}(M/x)$ is 3-connected and has an N-minor;
(ii) there is a four-element fan (x_1, x_2, x_3, x_4) of M such that $x_1, x_3 \in C^*$, and $\mathrm{si}(M/x_2)$ is 3-connected with an N-minor;
(iii) there is a segment-cosegment pair (L, L^*) such that $L \subseteq C^*$, and $\mathrm{cl}(L) - L$ contains a single element e. In this case $e \notin C^*$ and $\mathrm{si}(M/e)$ is 3-connected with an N-minor. Moreover $M/\mathrm{cl}(L)$ is 3-connected with an N-minor, and if $x_i \in L$ then M/x_i is 3-connected up to a unique spore $(\mathrm{cl}(L) - x_i, y_i)$; or,
(iv) there is a segment-cosegment pair (L, L^*) such that L is a flat and $|L - C^*| \leq 1$. In this case M/L is 3-connected with an N-minor, and if $x_i \in L$ then M/x_i is 3-connected up to a unique spore $(L - x_i, y_i)$.

We note that if (L, L^*) is a segment-cosegment pair of the matroid M, and $M/\mathrm{cl}(L)$ has an N-minor, then $|E(M) - \mathrm{cl}(L)| \geq 4$. Under these hypotheses Proposition 3.6 tells us that $M/\mathrm{cl}(L)$ is isomorphic to $\mathrm{co}(\mathrm{si}(M/x_i))$ for any element $x_i \in L$. Therefore Theorem 1.1 does indeed follow from Theorem 1.2.

By dualizing we immediately obtain the following corollary of Theorem 1.1.

Theorem 1.3. *Suppose that M and N are 3-connected matroids such that $|E(N)| \geq 4$ and C is a circuit of M with the property that $M\backslash x_0$ has an N-minor for some $x_0 \in C$. Then either:*

(i) *there is an element* $x \in C$ *such that* $\mathrm{co}(M\backslash x)$ *is* 3*-connected and has an* N*-minor;*
(ii) *there is an element* $x \in C$ *such that* $\mathrm{si}(\mathrm{co}(M\backslash x))$ *is* 3*-connected and has an* N*-minor; or,*
(iii) *there is a four-element fan* (x_1, x_2, x_3, x_4) *in* M *such that* $x_2, x_4 \in C$*, and* $\mathrm{co}(M\backslash x_3)$ *is* 3*-connected with an* N*-minor.*

We note that Lemos [5] has considered the situation that a 3-connected matroid M contains a circuit C with the property that $M\backslash x$ is not 3-connected for any element $x \in C$. He shows that in this case C meets at least two triads of M.

In Section 2 we introduce essential notions of matroid connectivity. Section 3 contains a detailed discussion of one of the structures we uncover: segment-cosegment pairs. In Section 4 we collect some preliminary lemmas, and in Section 5 we complete the proof of Theorem 1.2. Notation and terminology generally follows that of Oxley [6], except that the simple (respectively cosimple) matroid associated with the matroid M is denoted $\mathrm{si}(M)$ (respectively $\mathrm{co}(M)$). We consistently write z instead of $\{z\}$ for the set containing the single element z.

2. Essentials

This section collects some elementary results on matroid connectivity. Let M be a matroid on the ground set E. The *connectivity function* of M, denoted by λ_M (or λ when there is no ambiguity), takes subsets of E to $\mathbb{Z}^+ \cup \{0\}$. It is defined so that

$$\lambda_M(X) = \mathrm{r}_M(X) + \mathrm{r}_M(E - X) - \mathrm{r}(M)$$

for any subset $X \subseteq E$. Note that $\lambda(X) = \lambda(E - X)$ and $\lambda_{M^*}(X) = \lambda_M(X)$ for any subset $X \subseteq E$. It is well known, and easy to verify, that the connectivity function of M is submodular. That is, for all $X, Y \subseteq E$, the inequality

$$\lambda(X \cap Y) + \lambda(X \cup Y) \leq \lambda(X) + \lambda(Y)$$

is satisfied.

We say that a subset $X \subseteq E$ is *k-separating* or a *k-separator* of M if $\lambda(X) < k$, and we say that a partition $(X, E - X)$ is a *k-separation* of M if X is k-separating and $|X|, |E - X| \geq k$. A k-separator X or a k-separation $(X, E - X)$ is *exact* if $\lambda(X) = k - 1$. A matroid M is *n-connected* if M has no k-separation for any $k < n$. We define a *k-partition* of M to be a partition $(X_1, X_2, \ldots, X_n)$ of E such that X_i is k-separating for all $1 \leq i \leq n$. We say that the k-partition $(X_1, X_2, \ldots, X_n)$ is *exact* if each k-separator X_i is exact.

The next result is easy.

Proposition 2.1. *Let N be a minor of the matroid M and let X be a subset of $E(M)$. Then $\lambda_N(E(N) \cap X) \leq \lambda_M(X)$.*

Proposition 2.2. *Suppose that M is a matroid and that (X, Y, z) is a partition of $E(M)$. If $\lambda(X) = \lambda(Y)$ then z is in $\mathrm{cl}(X) \cap \mathrm{cl}(Y)$ or in $\mathrm{cl}^*(X) \cap \mathrm{cl}^*(Y)$, but not both.*

Proof. Since

$$\lambda(X) = \mathrm{r}(X) + \mathrm{r}(Y \cup z) - \mathrm{r}(M) = \mathrm{r}(X \cup z) + \mathrm{r}(Y) - \mathrm{r}(M) = \lambda(Y)$$

it follows that $\mathrm{r}(Y \cup z) - \mathrm{r}(Y) = \mathrm{r}(X \cup z) - \mathrm{r}(X)$. Therefore, $z \in \mathrm{cl}(X)$ if and only if $z \in \mathrm{cl}(Y)$. In the case that $z \notin \mathrm{cl}(X)$ and $z \notin \mathrm{cl}(Y)$ then

$$\begin{aligned}\mathrm{r}^*(Y \cup z) - \mathrm{r}^*(Y) = (&|Y \cup z| + \mathrm{r}(X) - \mathrm{r}(M)) \\ &- (|Y| + \mathrm{r}(X \cup z) - \mathrm{r}(M)) = 1 + \mathrm{r}(X) - \mathrm{r}(X \cup z) = 0.\end{aligned}$$

Thus $z \in \mathrm{cl}^*(Y)$. The same argument shows that $z \in \mathrm{cl}^*(X)$.

Finally we note that $z \in \mathrm{cl}^*(X)$ if and only if $z \notin \mathrm{cl}(Y)$. Thus $\mathrm{cl}(X) \cap \mathrm{cl}(Y)$ and $\mathrm{cl}^*(X) \cap \mathrm{cl}^*(Y)$ are disjoint. □

The next result is well known, and follows without difficulty from the dual of [8, Lemma 2.5].

Proposition 2.3. *Suppose that X is an exactly 3-separating set of the 3-connected matroid M. Suppose also that $A \subseteq E(M) - X$. If $|A| \geq 3$ and $A \subseteq \mathrm{cl}^*(X)$ then A is a cosegment of M.*

Definition 2.4. Suppose that M is a matroid and that $x \in E(M)$. Let (X_1, X_2) be a partition of $E(M) - x$ such that there is a positive integer k with the property that:

(i) $\lambda(X_1) = \lambda(X_2) = k - 1$;
(ii) $\mathrm{r}(X_1), \mathrm{r}(X_2) \geq k$; and,
(iii) $x \in \mathrm{cl}(X_1) \cap \mathrm{cl}(X_2)$.

In this case (X_1, X_2, x) is a *vertical k-partition* of M.

The next result is well known and easy to prove.

Proposition 2.5. *Let M be a 3-connected matroid and suppose that $\mathrm{si}(M/x)$ is not 3-connected for some $x \in E(M)$. Then there exists a vertical 3-partition (X_1, X_2, x) of M.*

Proposition 2.6. *Suppose that (X_1, X_2, x) is vertical k-partition of the k-connected matroid M. Let A be a subset of $\mathrm{cl}(X_2 \cup x)$. Then $(X_1 - A, (X_2 \cup A) - x, x)$ is also a vertical k-partition of M.*

Proof. Suppose that z is some element in $X_1 \cap A$. Then $\lambda(X_1 - z)$ is either $k-2$ or $k-1$. If $\lambda(X_1 - z) = k-2$ then $(X_1 - z,\, X_2 \cup \{x,\, z\})$ is a $(k-1)$-separation of M, a contradiction. Hence $\lambda(X_1 - z) = k-1$ which implies that $\mathrm{r}(X_1 - z) = \mathrm{r}(X_1)$. Thus $\mathrm{cl}(X_1 - z) = \mathrm{cl}(X_1)$, and hence $x \in \mathrm{cl}(X_1 - z)$. It follows that $(X_1 - z,\, X_2 \cup z,\, x)$ is a vertical k-partition of M. By continuing to transfer elements in $X_1 \cap A$ from X_1 into X_2 we eventually conclude that $(X_2 - A,\, (X_2 \cup A) - x,\, x)$ is a vertical k-partition of M, as desired. □

Suppose that M_1 and M_2 are matroids such that $E(M_1) \cap E(M_2) = \{p\}$. Then we can define the *parallel connection* of M_1 and M_2, denoted by $P(M_1,\, M_2)$. The ground set of $P(M_1,\, M_2)$ is $E(M_1) \cup E(M_2)$. If p is a loop in neither M_1 nor M_2 then the circuits of $P(M_1,\, M_2)$ are exactly the circuits of M_1, the circuits of M_2, and sets of the form $(C_1 - p) \cup (C_2 - p)$, where C_i is a circuit of M_i such that $p \in C_i$ for $i = 1,\, 2$. If p is a loop in M_1 then $P(M_1,\, M_2)$ is defined to be the direct sum of M_1 and M_2/p. Similarly, if p is a loop in M_2 then $P(M_1,\, M_2)$ is defined to be the direct sum of M_1/p and M_2. We say that p is the *basepoint* of the parallel connection. It is clear that $P(M_1,\, M_2) = P(M_2,\, M_1)$.

The next result follows from [6, Proposition 7.1.15 (v)].

Proposition 2.7. *Suppose that M_1 and M_2 are matroids such that $E(M_1) \cap E(M_2) = \{p\}$. If $e \in E(M_1) - p$ then $P(M_1,\, M_2)\backslash e = P(M_1\backslash e,\, M_2)$ and $P(M_1,\, M_2)/e = P(M_1/e,\, M_2)$.*

Assume that M_1 and M_2 are matroids such that $E(M_1) \cap E(M_2) = \{p\}$. If p is not a loop or a coloop in either M_1 or M_2 then $P(M_1,\, M_2)\backslash p$ is the 2-*sum* of M_1 and M_2, denoted by $M_1 \oplus_2 M_2$. We say that p is the *basepoint* of the 2-sum.

The next result follows from [10, (2.6)].

Proposition 2.8. *If $(X_1,\, X_2)$ is an exact 2-separation of a matroid M then there exist matroids M_1 and M_2 on the ground sets $X_1 \cup p$ and $X_2 \cup p$ respectively, where p is in neither X_1 nor X_2, such that M is equal to $M_1 \oplus_2 M_2$.*

Proposition 2.9. *Suppose that N is a 3-connected matroid. Let M be a matroid with a vertical 3-partition $(X_1,\, X_2,\, x)$ such that N is a minor of M/x. Then either $|E(N) \cap X_1| \leq 1$, or $|E(N) \cap X_2| \leq 1$.*

Proof. Since $(X_1,\, X_2)$ is a 2-separation of M/x the result follows immediately from Proposition 2.1. □

Lemma 2.10. *Suppose that N is a 3-connected matroid such that $|E(N)| \geq 2$. Let M be a matroid with a vertical 3-partition $(X_1,\, X_2,\, x)$*

such that N is a minor of M/x. If $|E(N) \cap X_1| \leq 1$ then $M/x/e$ has an N-minor for every element $e \in X_1 - \mathrm{cl}_M(X_2)$.

Proof. Since (X_1, X_2) is an exact 2-separation of M/x, it follows from Proposition 2.8 that M/x is the 2-sum of matroids M_1 and M_2 along the basepoint p, where $E(M_1) = X_1 \cup p$ and $E(M_2) = X_2 \cup p$. Thus $M/x = P(M_1, M_2)\backslash p$.

Suppose that $E(N) \cap X_1 = \emptyset$. Then there is a partition (A, B) of X_1 such that N is a minor of $M/x/A\backslash B$. Suppose that p is a loop in $M_1/A\backslash B$. Proposition 2.7 implies that

$$M/x/A\backslash B = P(M_1/A\backslash B, M_2)\backslash p.$$

Now the definition of parallel connection implies that $M/x/A\backslash B$ is isomorphic to M_2/p. It is easily seen that if $e \in X_1$ then there is a minor M' of M_1/e such that $E(M') = \{p\}$ and p is a loop of M'. Proposition 2.7 implies that $P(M', M_2)\backslash p$ is a minor of $M/x/e$. But $P(M', M_2)\backslash p$ is isomorphic to M_2/p, so $M/x/e$ has an N-minor.

Next we suppose that p is a coloop of $M_1/A\backslash B$. Then, by definition of the parallel connection, $M/x/A\backslash B$ is isomorphic to $M_2\backslash p$. Suppose that $e \in X_1 - \mathrm{cl}(X_2)$. Since p is not a coloop of M_2 it follows easily that $p \in \mathrm{cl}_M(X_2)$. Thus e is not parallel to p in M_1. Therefore there is a minor M' of M_1/e such that $E(M') = \{p\}$ and p is a coloop of M'. Again using Proposition 2.7 we see that $P(M', M_2)\backslash p$ is a minor of $M/x/e$. But since $P(M', M_2)\backslash p$ is isomorphic to $M_2\backslash p$ we deduce that $M/x/e$ has an N-minor.

Now we assume that $|E(N) \cap X_1| = 1$ and that z is the unique element in $E(N) \cap X_1$. There is a partition (A, B) of $X_1 - z$ such that N is a minor of $M/x/A\backslash B$. It follows from Proposition 2.7 that $P(M_1/A\backslash B, M_2)\backslash p$ has an N-minor. Consider the matroid $M_1/A\backslash B$. If $\{z, p\}$ is not a parallel pair in this matroid then z must be a loop or coloop in $P(M_1/A\backslash B, M_2)\backslash p$. This implies that z is a loop or coloop in N, a contradiction as N is 3-connected and $|E(N)| \geq 2$. Therefore z and p are parallel in $M_1/A\backslash B$, and therefore $P(M_1/A\backslash B, M_2)\backslash p$ is isomorphic to M_2. Thus M_2 has an N-minor.

Since p is not a loop or coloop of M_1 there is a circuit of size at least two in M_1 that contains p. Suppose that $e \in X_1 - \mathrm{cl}_M(X_2)$. Then e cannot be parallel to p in M_1, so M_1/e has a circuit of size at least two that contains p. Hence there is a minor M' of M_1/e such that $p \in E(M')$ and M' consists of a parallel pair. Proposition 2.7 implies that $P(M', M_2)\backslash p$ is a minor of $M/x/e$. But $P(M', M_2)\backslash p$ is isomorphic to M_2, so $M/x/e$ has an N-minor. □

Definition 2.11. Suppose that M is a matroid and that A and B are subsets of $E(M)$. The *local connectivity* between A and B, denoted by $\sqcap(A, B)$, is defined to be $\mathrm{r}(A)+\mathrm{r}(B)-\mathrm{r}(A\cup B)$. Equivalently, $\sqcap(A, B)$ is equal to $\lambda_{M|(A\cup B)}(A)$.

Proposition 2.12. [8, Lemma 2.4 (iv)] *Let M be a matroid and let (A, B, C) be a partition of $E(M)$. Then $\sqcap(A, B)+\lambda(C)=\sqcap(A, C)+\lambda(B)$. Hence $\sqcap(A, B)=\sqcap(A, C)$ if and only if $\lambda(B)=\lambda(C)$.*

Corollary 2.13. *Let (X, Y, Z) be an exact 3-partition of the 3-connected matroid M. Then $\sqcap(X, Y)=\sqcap(X, Z)=\sqcap(Y, Z)$.*

Proposition 2.14. *Suppose that M is a matroid and that X and Y are disjoint subsets of $E(M)$ such that $\sqcap(X, Y)=1$. If $x, y\in X\cap \mathrm{cl}(Y)$ then $\mathrm{r}(\{x, y\})\leq 1$.*

Proof. Assume that $\mathrm{r}(\{x, y\})=2$. Let $X'=\mathrm{cl}(X)$ and $Y'=\mathrm{cl}(Y)$. It is easy to see that $\mathrm{r}(X'\cup Y')=\mathrm{r}(X\cup Y)$. However

$$\mathrm{r}(X'\cup Y')\leq \mathrm{r}(X')+\mathrm{r}(Y')-\mathrm{r}(X'\cap Y')\leq \mathrm{r}(X)+\mathrm{r}(Y)-2=\mathrm{r}(X\cup Y)-1.$$

This contradiction completes the proof. □

We conclude this section by stating a fundamental tool in the study of 3-connected matroids, due to Bixby [1].

Theorem 2.15 (Bixby's Lemma)**.** *Let M be a 3-connected matroid and suppose that x is an element of $E(M)$. Then either $\mathrm{si}(M/x)$ or $\mathrm{co}(M\backslash x)$ is 3-connected.*

3. Segment-cosegment pairs

Suppose that M is a matroid. Recall that L is a segment of M if $|L|\geq 3$ and every three-element subset of L is a circuit of M, and that L^* is a cosegment of M if $|L^*|\geq 3$ and every three-element subset of L^* is a cocircuit. We restate the definition of segment-cosegment pairs given in Section 1.

Definition 3.1. Suppose that $L=\{x_1,\ldots,x_t\}$ is a segment of the matroid M and there is a set $L^*=\{y_1,\ldots,y_t\}$ with the property that $L\cap L^*=\emptyset$ and $(\mathrm{cl}(L)-x_i)\cup y_i$ is a cocircuit of M for all $i\in\{1,\ldots,t\}$. In this case we say that (L, L^*) is a *segment-cosegment pair* of M.

In a 3-connected matroid a segment-cosegment pair is an example of a 'crocodile', a structure that provides a collection of equivalent 3-separations. 'Crocodiles' were considered by Hall, Oxley, and Semple [4]. The next result explains the name segment-cosegment pair.

Proposition 3.2. *Suppose that $(L,\, L^*)$ is a segment-cosegment pair of the 3-connected matroid M. Then L^* is a cosegment of M.*

Proof. Suppose that $y_i \in L^*$. The definition of a segment-cosegment pair means that $y_i \in \mathrm{cl}^*(\mathrm{cl}(L))$. Thus $L^* \subseteq \mathrm{cl}^*(\mathrm{cl}(L))$. Moreover $\mathrm{cl}(L)$ is exactly 3-separating in M. The result follows by Proposition 2.3. □

Proposition 3.3. *Suppose that $(L,\, L^*)$ is a segment-cosegment pair of the 3-connected matroid M. Then $M/\mathrm{cl}(L)$ is 3-connected.*

Proof. Suppose that $L = \{x_1, \ldots, x_t\}$ and $L^* = \{y_1, \ldots, y_t\}$. Assume that $M/\mathrm{cl}(L)$ is not 3-connected, so that $(X_1,\, X_2)$ is a k-separation of $M/\mathrm{cl}(L)$ for some $k \leq 2$. Let $L_0 = \mathrm{cl}(L)$. Note that for $i \in \{1,\, 2\}$ we have

$$\mathrm{r}_{M/L_0}(X_i) = \mathrm{r}_M(X_i \cup L_0) - \mathrm{r}_M(L_0) = \mathrm{r}_M(X_i) - \sqcap_M(X_i,\, L_0),$$

so $\mathrm{r}_M(X_i) = \mathrm{r}_{M/L_0}(X_i) + \sqcap_M(X_i,\, L_0)$.

Suppose that $\sqcap_M(X_1,\, L_0) = 0$. Then $\mathrm{r}_M(X_1) = \mathrm{r}_{M/L_0}(X_1)$ and $\mathrm{r}_M(X_2 \cup L_0) = \mathrm{r}_{M/L_0}(X_2) + 2$, so

$$\begin{aligned}\lambda_M(X_1) = \mathrm{r}_{M/L_0}(X_1) + (\mathrm{r}_{M/L_0}(X_2) + 2) - (\mathrm{r}(M/L_0) + 2)&\\ = \lambda_{M/L_0}(X_1) < k.&\end{aligned}$$

This is a contradiction as M is 3-connected. By using a symmetric argument we can conclude that $\sqcap_M(X_i,\, L_0) > 0$ for all $i \in \{1,\, 2\}$.

Suppose that $x_i \in \mathrm{cl}_M(X_1)$ for some $i \in \{1, \ldots, t\}$. Then there is a circuit $C_1 \subseteq X_1 \cup x_i$ such that $x_i \in C_1$. For all $k \in \{1, \ldots, t\} - i$ the set $(L_0 - x_k) \cup y_k$ is a cocircuit. It cannot be the case that C_1 meets this cocircuit in a single element, so $y_k \in X_1$ for all $k \in \{1, \ldots, t\} - i$.

Now suppose that $x_j \in \mathrm{cl}_M(X_2)$ for some $j \in \{1, \ldots, t\}$. By using the same arguments as above we can conclude that $L^* - y_j \subseteq X_2$. As $L^* - y_i$ and $L^* - y_j$ have a non-empty intersection this is a contradiction. Therefore $\mathrm{cl}_M(X_2) \cap L = \emptyset$. Note that $\sqcap(X_2,\, L_0) \leq 2$ because $\mathrm{r}(L_0) = 2$. If $\sqcap(X_2,\, L_0)$ were two, it would follow that $L_0 \subseteq \mathrm{cl}(X_2)$. Hence $\sqcap(X_2,\, L_0) = 1$.

Let j be an element of $\{1, \ldots, t\} - i$. Then $L_0 \subseteq \mathrm{cl}_M(X_2 \cup x_j)$, and there must be a circuit $C_2 \subseteq X_2 \cup \{x_i,\, x_j\}$ such that $\{x_i,\, x_j\} \subseteq C_2$. But then C_2 meets the cocircuit $(L_0 - x_j) \cup y_j$ in a single element, x_i. From this contradiction we conclude that $\mathrm{cl}_M(X_1) \cap L = \emptyset$, and by symmetry $\mathrm{cl}_M(X_2) \cap L = \emptyset$. This means that

$$\sqcap_M(X_1,\, L_0) = \sqcap_M(X_2,\, L_0) = 1.$$

It must be the case that $x_2 \in \mathrm{cl}_M(X_1 \cup x_1)$, and there is a circuit $C_3 \subseteq X_1 \cup \{x_1,\, x_2\}$ such that $\{x_1,\, x_2\} \subseteq C_3$. Since $(L_0 - x_1) \cup y_1$ is a cocircuit we conclude that $y_1 \in X_1$. But we can use an identical

argument to show that $y_1 \in X_2$. This contradiction completes the proof. $\square$

We now restate the definition of a spore.

Definition 3.4. Suppose that P is a rank-one flat of a matroid M and that s is an element of $E(M)$ such that $P \cup s$ is a cocircuit. Then we say that (P, s) is a *spore*.

Recall from Section 1 that a matroid M is 3-connected up to a unique spore if it contains a single spore (P, s), and whenever (X, Y) is a k-separation of M for some $k < 3$ then either $X \subseteq P \cup s$ or $Y \subseteq P \cup s$.

Lemma 3.5. *Suppose that (L, L^*) is a segment-cosegment pair of the 3-connected matroid M where $|E(M) - \mathrm{cl}(L)| \geq 4$. Let $L = \{x_1, \ldots, x_t\}$ and $L^* = \{y_1, \ldots, y_t\}$. Then M/x_i is 3-connected up to a unique spore $(\mathrm{cl}(L) - x_i, y_i)$, for all $i \in \{1, \ldots, t\}$.*

Proof. Let E be the ground set of M and let $L_0 = \mathrm{cl}(L)$. We will show that M/x_i is 3-connected up to the unique spore $(L_0 - x_i, y_i)$. Certainly $(L_0 - x_i, y_i)$ is a spore of M/x_i. Suppose that (P, s) is a spore of M/x_i that is distinct from $(L_0 - x_i, y_i)$.

We initially assume that $L_0 - x_i = P$. Thus $s \neq y_i$. As $(L_0 - x_i) \cup s$ and $(L_0 - x_i) \cup y_i$ are both cocircuits of M/x_i it follows that $E - (L_0 \cup \{s, y_i\})$ is the intersection of two hyperplanes of M/x_i. Thus

$$\mathrm{r}_{M/x_i}(E - (L_0 \cup \{s, y_i\})) \leq \mathrm{r}(M/x_i) - 2.$$

and therefore

$$\mathrm{r}_{M/L_0}(E - (L_0 \cup \{s, y_i\})) \leq \mathrm{r}(M/x_i) - 2 = \mathrm{r}(M/L_0) - 1.$$

Hence $\{s, y_i\}$ contains a cocircuit in M/L_0. Therefore M/L_0 contains a cocircuit of size at most two, a contradiction as M/L_0 is 3-connected by Proposition 3.3, and $|E(M/L_0)| \geq 4$.

Now we must assume that $L_0 - x_i \neq P$. Hence $P \cup x_i$ is a rank-two flat of M that meets L_0 in exactly one element, x_i. Suppose that P contains a single element p. Then $\{p, s\}$ is a cocircuit of M, a contradiction. Therefore $P \cup x_i$ contains at least one triangle. Suppose that P does not contain y_j, where $j \neq i$. Then there is a triangle in $P \cup x_i$ that meets the cocircuit $(L_0 - x_j) \cup y_j$ in exactly one element, x_i. This contradiction shows that $L^* - y_i \subseteq P$.

Assume that $t > 3$. As L^* is a cosegment there is a triad of M contained in $L^* - y_i$. However this triad is also contained in the segment $P \cup x_i$, and is therefore a triangle. But $|E(M)| > 4$ and a 3-connected matroid with at least five elements cannot contain a triangle that is also a triad. This contradiction shows that $t = 3$.

Suppose $j \in \{1, 2, 3\}$ and that $j \neq i$. If $|P| > 2$ then there is a triangle contained in P that contains y_j. However this triangle would meet the cocircuit $(L_0 - x_j) \cup y_j$ in exactly one element. Thus $|P| = 2$, and $P = L^* - y_i$.

Suppose that $j, k \in \{1, 2, 3\}$ and neither j nor k is equal to i. Then $L_0 \cup P$ contains the two cocircuits $(L_0 - x_j) \cup y_j$ and $(L_0 - x_k) \cup y_k$. Hence $\mathrm{r}_M(E - (L_0 \cup P)) \leq \mathrm{r}(M) - 2$. However it is easy to see that $\mathrm{r}_M(L_0 \cup P) = 3$. As $|P| = 2$ it follows that $E - (L_0 \cup P)$ contains at least two elements. Thus $(L_0 \cup P, E - (L_0 \cup P))$ is a 2-separation of M, a contradiction.

We have shown that $(L_0 - x_i, y_i)$ is the unique spore of M/x_i. Next we show that M/x_i is 3-connected up to this spore. Suppose that (X, Y) is a k-separation of M/x_i for some $k < 3$. By relabeling if necessary we will assume that $y_i \in X$. Assume that the result is false, so that neither X nor Y is contained in $(L_0 - x_i) \cup y_i$. Therefore X contains at least one element from $E - (L_0 \cup y_i)$. As M/L_0 is 3-connected by Proposition 3.3 we deduce from Proposition 2.1 that either $X - L_0$ or $Y - L_0$ contains at most one element. We have already concluded that $X - L_0$ contains at least two elements (as $y_i \in X$), so $Y - L_0$ contains precisely one element. As M is 3-connected it contains no parallel pairs, so M/x_i contains no loops. Therefore $\mathrm{r}_{M/x_i}(Y) = 2$, and hence $\mathrm{r}_{M/x_i}(X) \leq \mathrm{r}(M/x_i) - 1$. Thus Y contains a cocircuit of M/x_i. As M/x_i has no coloops, and any cocircuit that meets a parallel class contains that parallel class it follows that $L_0 - x_i \subseteq Y$. Let s be the single element in $Y - L_0$. It cannot be the case that Y is a cocircuit in M/x_i, for that would imply that $(L_0 - x_i, s)$ is a spore of M/x_i that differs from $(L_0 - x_i, y_i)$, contradicting our earlier conclusion. Now we see that $Y - s = L_0 - x_i$ must be a cocircuit of M/x_i, but this is a contradiction as $L_0 - x_i$ is properly contained in the cocircuit $(L_0 - x_i) \cup y_i$. The completes the proof. $\square$

The next result shows that Theorem 1.1 is a consequence of Theorem 1.2.

Proposition 3.6. *Suppose that (L, L^*) is a segment-cosegment pair of a matroid M, and that $M/\operatorname{cl}(L)$ is 3-connected and $|E(M) - \operatorname{cl}(L)| \geq 4$. Let $L = \{x_1, \ldots, x_t\}$ and $L^* = \{y_1, \ldots, y_t\}$. Then $\operatorname{co}(\operatorname{si}(M/x_i)) \cong M/\operatorname{cl}(L)$ for any element $x_i \in L$.*

Proof. Let $L_0 = \operatorname{cl}(L)$ and let $x_j \neq x_i$ be an element of L. Suppose that P and S are disjoint subsets of $E(M) - x_i$ chosen so that $\operatorname{co}(\operatorname{si}(M/x_i)) \cong M/x_i \backslash P/S$. As $L_0 - x_i$ is a parallel class in M/x_i we may assume that $L_0 - \{x_i, x_j\} \subseteq P$ and that $x_j \notin P$. We may assume that $y_i \notin P$,

and hence $\{x_j, y_i\}$ is a union of cocircuits in $M/x_i\backslash P$. Therefore we may assume $x_j \in S$. Since the elements in $L_0 - \{x_i, x_j\}$ are loops in $M/x_i/x_j$ it follows that

$$M/x_i\backslash P/S = M/x_i/x_j/(L_0-\{x_i, x_j\})\backslash(P-(L_0-\{x_i, x_j\}))/(S-x_j).$$

This last matroid is equal to $M/L_0\backslash(P-(L_0-\{x_i, x_j\}))/(S-x_j)$. Since M/L_0 is 3-connected and the elements in $P-(L_0-\{x_i, x_j\})$ are either loops or parallel elements in M/L_0 it follows that $P = L_0 - \{x_i, x_j\}$. Thus $M/x_i\backslash P/S = M/L_0/(S-x_j)$. But M/L_0 is 3-connected, so $S-x_j$ must be empty. Thus $M/L_0 \cong \mathrm{co}(\mathrm{si}(M/x_i))$, as desired. □

4. Preliminary lemmas

Proposition 4.1. *Suppose that C^* is a cocircuit of the 3-connected matroid M. Assume that (X_1, X_2, x) is a vertical 3-partition of M such that $x \in C^*$. Then $C^* \cap (X_1 - \mathrm{cl}(X_2)) \neq \emptyset$ and $C^* \cap (X_2 - \mathrm{cl}(X_1)) \neq \emptyset$.*

Proof. Note that $\mathrm{r}(X_1)$, $\mathrm{r}(X_2) \geq 3$ implies that $|E(M)| \geq 4$, so every circuit and cocircuit of M contains at least three elements. Let X be $X_1 - \mathrm{cl}(X_2)$. The fact that $\mathrm{r}(X_1) \geq 3$ implies that X contains a cocircuit, so $|X| \geq 3$. Suppose that x is not in $\mathrm{cl}(X)$. Then $\mathrm{r}(X) < \mathrm{r}(X_1)$. Since $|X| \geq 3$ this implies that $(X, \mathrm{cl}(X_2))$ is a 2-separation of M, a contradiction.

Now suppose that $C^* \subseteq \mathrm{cl}(X_2)$. Then as $x \in \mathrm{cl}(X)$ and $x \in C^*$ there is a circuit in M that meets C^* in exactly one element, x. This is a contradiction. The same argument shows that $C^* \cap (X_2 - \mathrm{cl}(X_1)) \neq \emptyset$, so the proposition holds. □

Definition 4.2. Suppose that M is a 3-connected matroid and that A is a subset of $E(M)$. A *minimal partition* with respect to A is a vertical 3-partition (X_1, X_2, x) of M that satisfies the following properties:

(i) $x \in A$;
(ii) if (Y_1, Y_2, y) is a vertical 3-partition of M such that $y \in A \cap (X_1 \cup x)$ and $X_2 \cap Y_1 = \emptyset$, then $(Y_1, Y_2, y) = (X_1, X_2, x)$; and,
(iii) if (Y_1, Y_2, y) is a vertical 3-partition of M such that $y \in A \cap (X_1 \cup x)$ and $X_2 \cap Y_2 = \emptyset$ then $(Y_2, Y_1, y) = (X_1, X_2, x)$.

If there is no ambiguity we will refer to a minimal partition with respect to A as a *minimal partition.*

Lemma 4.3. *Suppose that M is a 3-connected matroid and that A is a subset of $E(M)$. Suppose that for some element $z \in A$ there is a vertical 3-partition (Z_1, Z_2, z) of M. Let $Z = Z_1 - \mathrm{cl}(Z_2)$. Then there is a minimal partition (X_1, X_2, x) with respect to A such that $X_1 \subseteq Z$ and $x \in A \cap (Z \cup z)$.*

Proof. Let $\mathcal{Z}$ be the family of vertical 3-partitions (S_1, S_2, z) with the property that $S_1 \subseteq Z_1$. Choose (Z_1', Z_2', z) from $\mathcal{Z}$ so that if (S_1, S_2, z) is in $\mathcal{Z}$, then S_1 is not properly contained in Z_1'. Observe that Proposition 2.6 implies that $Z_1' \subseteq Z$.

Let $\mathcal{S}$ be the family of vertical 3-partitions (S_1, S_2, s) with $s \in A \cap (Z_1' \cup z)$. Let $\mathcal{S}_0$ be the set of vertical 3-partitions (S_1, S_2, s) in $\mathcal{S}$ with the property that either $S_1 \subseteq Z_1'$ or $S_2 \subseteq Z_1'$. Without loss of generality we will assume that if (S_1, S_2, s) is in $\mathcal{S}_0$ then $S_1 \subseteq Z_1'$. Suppose that (S_1, S_2, z) is a member of $\mathcal{S}_0$. Then our choice of (Z_1', Z_2', z) means that $S_1 = Z_1'$ and $S_2 = Z_2'$. If (Z_1', Z_2', z) is the only member of $\mathcal{S}_0$ then we can set (X_1, X_2, x) to be (Z_1', Z_2', z), and we will be done. Therefore we will assume that there is at least one vertical 3-partition (S_1, S_2, s) in $\mathcal{S}_0$ such that $s \neq z$. Let $\mathcal{S}_1$ be the collection of such partitions.

We now let (X_1, X_2, x) be a vertical 3-partition in $\mathcal{S}_1$ chosen so that if $(S_1, S_2, s) \in \mathcal{S}_1$, then $S_1 \cup s$ is not properly contained in $X_1 \cup x$. We will prove that (X_1, X_2, x) is the desired vertical 3-partition.

It is certainly true that $X_1 \subseteq Z$. If there is some element e in $X_1 \cap \mathrm{cl}(X_2 \cup x)$ then $(X_1 - e, X_2 \cup e, x)$ is a vertical 3-partition by Proposition 2.6. However this contradicts our choice of (X_1, X_2, x). Therefore $X_2 \cup x$ is a flat. We assume that (Y_1, Y_2, y) is a vertical 3-partition and that $y \in A \cap (X_1 \cup x)$. As $X_1 \subseteq Z_1'$ it follows that $y \in A \cap Z_1'$. Our assumption on (X_1, X_2, x) means that neither $Y_1 \cup y$ nor $Y_2 \cup y$ can be properly contained in $X_1 \cup x$.

Suppose that $X_2 \cap Y_1 = \emptyset$. Then $Y_1 \cup y$ must be equal to $X_1 \cup x$. If $y \neq x$ then the fact that $y \in \mathrm{cl}(Y_2)$ and $Y_2 = X_2$ means that $y \in \mathrm{cl}(X_2)$, which is a contradiction as $X_2 \cup x$ is a flat. Therefore $y = x$, so (Y_1, Y_2, y) is equal to (X_1, X_2, x). The same argument shows that if $X_2 \cap Y_2 = \emptyset$ then $(Y_1, Y_2, y) = (X_2, X_1, x)$. Thus (X_1, X_2, x) is the desired minimal partition. □

Proposition 4.4. *Suppose that M is a matroid and that $A \subseteq E(M)$. Suppose that (X_1, X_2, x) is a minimal partition with respect to A. Then $X_2 \cup x$ is a flat of M.*

Proof. Suppose that there is some element $z \in X_1 \cap \mathrm{cl}(X_2 \cup x)$. Then $(X_1 - z, X_2 \cup z, x)$ is a vertical 3-partition of M by Proposition 2.6. This contradicts the fact that (X_1, X_2, x) is a minimal partition. □

Lemma 4.5. *Suppose that M is a 3-connected matroid and that $A \subseteq E(M)$. Suppose that (X_1, X_2, x) is a minimal partition with respect to A. Suppose also that (Y_1, Y_2, y) is a vertical 3-partition of M such that $y \in A \cap X_1$ and $x \in Y_1$. Then the following statements hold:*

(i) $X_i \cap Y_j \neq \emptyset$ *for all* $i, j \in \{1, 2\}$*;*
(ii) *Each of* $X_1 \cap Y_2$, $(X_1 \cap Y_2) \cup y$, $X_2 \cap Y_1$, $(X_2 \cap Y_1) \cup x$, *and* $X_2 \cap Y_2$ *is* 3*-separating in* M*;*
(iii) $(X_1 \cap Y_1) \cup \{x, y\}$ *is* 4*-separating in* M*;*
(iv) *Neither* $X_1 \cap Y_1$ *nor* $X_1 \cap Y_2$ *is contained in* $\mathrm{cl}(X_2)$, $X_1 \cap Y_1 \nsubseteq \mathrm{cl}(Y_2)$, *and* $X_1 \cap Y_2 \nsubseteq \mathrm{cl}(Y_1)$*;*
(v) $\mathrm{r}((X_1 \cap Y_2) \cup y) = 2$*; and,*
(vi) *If* $(X_1 \cap Y_1) \cup \{x, y\}$ *is* 3*-separating in* M*, then* $\mathrm{r}((X_1 \cap Y_1) \cup \{x, y\}) = 2$.

Proof. We start by proving (i). Since $y \neq x$ the definition of a minimal partition means that $X_2 \cap Y_1 \neq \emptyset$ and $X_2 \cap Y_2 \neq \emptyset$. Moreover $X_2 \cup x$ is a flat of M by Proposition 4.4, and $y \in X_1$, so $y \notin \mathrm{cl}(X_2 \cup x)$. However $y \in \mathrm{cl}(Y_1) \cap \mathrm{cl}(Y_2)$. It follows that neither Y_1 nor Y_2 can be contained in $X_2 \cup x$. Thus both Y_1 and Y_2 meet X_1.

Next we prove (ii). Consider $X_1 \cap Y_2$. Since $\lambda(X_1) = 2$ and $\lambda(Y_2) = 2$ the submodularity of the connectivity function implies that $\lambda(X_1 \cap Y_2) + \lambda(X_1 \cup Y_2) \leq 4$. If $X_1 \cap Y_2$ is not 3-separating then $\lambda(X_1 \cup Y_2) \leq 1$. However $|X_1 \cup Y_2| \geq 2$ and the complement of $X_1 \cup Y_2$ certainly contains at least two elements, since it contains x, and $X_2 \cap Y_1$ is non-empty. Thus M has a 2-separation, a contradiction. This shows that $X_1 \cap Y_2$ is 3-separating.

Since X_1 and $Y_2 \cup y$ are both 3-separating the same argument shows that $(X_1 \cap Y_2) \cup y$ is 3-separating. Since the complement of $X_2 \cup Y_1$ contains both y and at least one element in $X_1 \cap Y_2$, we can also show that $X_2 \cap Y_1$ and $(X_2 \cap Y_1) \cup x$ are both 3-separating. The same argument shows that $X_2 \cap Y_2$ is 3-separating.

Consider (iii). The submodularity of the connectivity function shows that

$$\lambda((X_1 \cap Y_1) \cup \{x, y\}) + \lambda(X_1 \cup Y_1) \leq 4.$$

Thus if $(X_1 \cap Y_1) \cup \{x, y\}$ is not 4-separating then $\lambda(X_1 \cup Y_1) = 0$. But this cannot occur as $X_1 \cup Y_1$ is non-empty, and its complement contains $X_2 \cap Y_2$, which is non-empty.

Next we move to (iv). Since $X_2 \cup x$ is a flat of M it follows that $\mathrm{cl}(X_2)$ does not meet X_1. Therefore $\mathrm{cl}(X_2)$ cannot contain $X_1 \cap Y_1$ or $X_1 \cap Y_2$.

Suppose that $X_1 \cap Y_1$ is contained in $\mathrm{cl}(Y_2)$. Then $Y_1 - \mathrm{cl}(Y_2)$ is contained in $X_2 \cup x$. However Proposition 2.6 says that

$$(Y_1 - \mathrm{cl}(Y_2), \mathrm{cl}(Y_2) - y, y)$$

is a vertical 3-partition of M. Thus y is in the closure of $Y_1 - \mathrm{cl}(Y_2)$, which means that $y \in \mathrm{cl}(X_2 \cup x)$. But this is a contradiction as $y \in X_1$,

and $X_2 \cup x$ is a flat of M. The same argument shows that $X_1 \cap Y_2$ is not contained in $\mathrm{cl}(Y_1)$.

To prove (v) we suppose that $\mathrm{r}((X_1 \cap Y_2) \cup y) \geq 3$. Consider the partition $(X_1 \cap Y_2,\, X_2 \cup Y_1,\, y)$ of $E(M)$. It follows from (ii) that

$$\lambda((X_1 \cap Y_2) \cup y) = \lambda(X_1 \cap Y_2) = 2,$$

so $\lambda(X_2 \cup Y_1) = 2$. Furthermore $y \in \mathrm{cl}(Y_1)$, so y is in the closure of $X_2 \cup Y_1$. Proposition 2.2 shows that $y \in \mathrm{cl}(X_1 \cap Y_2)$, so $\mathrm{r}(X_1 \cap Y_2) \geq 3$. Now it is easy to see that

$$(X_1 \cap Y_2,\, X_2 \cup Y_1,\, y)$$

is a vertical 3-partition of M. However $y \in A \cap X_1$ and $X_1 \cap Y_2$ does not meet X_2, so we have a contradiction to the fact that $(X_1,\, X_2,\, x)$ is a minimal partition.

We conclude by proving (vi). Suppose that $\lambda((X_1 \cap Y_1) \cup \{x,\, y\}) = 2$. This implies that $\lambda(X_2 \cup Y_2) = 2$. Since $y \in \mathrm{cl}(Y_2)$ it follows easily that $\lambda((X_1 \cap Y_1) \cup x) = 2$. Consider the partition

$$((X_1 \cap Y_1) \cup x,\, X_2 \cup Y_2,\, y)$$

of $E(M)$. Since $y \in \mathrm{cl}(Y_2)$ it follows from Proposition 2.2 that y is in the closure of $(X_1 \cap Y_1) \cup x$. Thus if $\mathrm{r}((X_1 \cap Y_1) \cup \{x,\, y\}) \geq 3$ it follows that $\mathrm{r}((X_1 \cap Y_1) \cup x) \geq 3$. In this case

$$((X_1 \cap Y_1) \cup x,\, X_2 \cup Y_2,\, y)$$

is vertical 3-partition of M that violates the fact that $(X_1,\, X_2,\, x)$ is a minimal partition. This completes the proof of the lemma. □

Proposition 4.6. *Suppose that $(X_1,\, X_2,\, x)$ is a minimal partition of the 3-connected matroid M with respect to the set $A \subseteq E(M)$. Assume that $(Y_1,\, Y_2,\, y)$ is a vertical 3-partition of M such that $y \in A \cap X_1$ and $x \in Y_1$. If $|X_1 \cap Y_2| \geq 2$ then*

$$\sqcap((X_1 \cap Y_1) \cup \{x,\, y\},\, X_1 \cap Y_2) = \sqcap((X_1 \cap Y_1) \cup y,\, X_1 \cap Y_2) = 1.$$

Proof. The hypotheses imply that $|E(M)| \geq 4$, so every circuit or cocircuit of M contains at least three elements. Let $\pi = \sqcap((X_1 \cap Y_1) \cup \{x,\, y\},\, X_1 \cap Y_2)$. We know from Lemma 4.5 (v) that $\mathrm{r}(X_1 \cap Y_2) \leq 2$. Therefore $\pi \leq 2$. On the other hand, since $|X_1 \cap Y_2| \geq 2$, the fact that $\mathrm{r}((X_1 \cap Y_2) \cup y) \leq 2$ implies that $y \in \mathrm{cl}(X_1 \cap Y_2)$. This in turn implies that $\pi \geq 1$.

Assume that $\pi = 2$. Then $X_1 \cap Y_2 \subseteq \mathrm{cl}((X_1 \cap Y_1) \cup \{x,\, y\})$. Since $x,\, y \in \mathrm{cl}(Y_1)$ this means that $X_1 \cap Y_2 \subseteq \mathrm{cl}(Y_1)$. But this contradicts (iv) of Lemma 4.5. Exactly the same argument shows that $\sqcap((X_1 \cap Y_1) \cup y,\, X_1 \cap Y_2) = 1$. □

Lemma 4.7. *Suppose that (X_1, X_2, x) is a minimal partition of the 3-connected matroid M with respect to the set $A \subseteq E(M)$. Assume that (Y_1, Y_2, y) is a vertical 3-partition of M such that $y \in A \cap X_1$ and $x \in Y_1$. If $|X_1 \cap Y_2| \geq 2$ then $y \in \mathrm{cl}((X_1 \cap Y_1) \cup x)$.*

Proof. The hypotheses imply that every circuit of M contains at least three elements. Since $|X_1 \cap Y_2| \geq 2$ it follows from Lemma 4.5 (v) implies that $y \in \mathrm{cl}(X_1 \cap Y_2)$. We assume that $y \notin \mathrm{cl}((X_1 \cap Y_1) \cup x)$. Since $X_1 \cap Y_1$ is non-empty by Lemma 4.5 (i) it follows that $|(X_1 \cap Y_1) \cup x| \geq 2$, so $\lambda((X_1 \cap Y_1) \cup x) \geq 2$. Furthermore $\lambda((X_1 \cap Y_1) \cup \{x, y\}) \leq 3$ by (iii) of Lemma 4.5. As $y \in \mathrm{cl}(Y_2)$ we deduce that

$$2 \leq \lambda((X_1 \cap Y_1) \cup x) < \lambda((X_1 \cap Y_1) \cup \{x, y\}) \leq 3.$$

Thus $\lambda((X_1 \cap Y_1) \cup x) = 2$. Moreover it follows from (ii) in Lemma 4.5 that $\lambda((X_1 \cap Y_2) \cup y) = 2$. Therefore

$$((X_1 \cap Y_1) \cup x,\ (X_1 \cap Y_2) \cup y,\ X_2)$$

is an exact 3-partition.

As $x \in \mathrm{cl}(X_2)$ it follows that $\sqcap((X_1 \cap Y_1) \cup x, X_2) \geq 1$. Now Corollary 2.13 implies that $\sqcap((X_1 \cap Y_2) \cup y, X_2) \geq 1$. But (iv) and (v) of Lemma 4.5 imply that $X_1 \cap Y_2 \not\subseteq \mathrm{cl}(X_2)$ and that $\mathrm{r}((X_1 \cap Y_2) \cup y) = 2$. We deduce that $\sqcap((X_1 \cap Y_2) \cup y, X_2) = 1$. Again using Corollary 2.13 we see that

$$\sqcap((X_1 \cap Y_1) \cup x,\ (X_1 \cap Y_2) \cup y) = 1.$$

Proposition 4.6 tells us that

$$\sqcap((X_1 \cap Y_1) \cup \{x, y\},\ X_1 \cap Y_2) = 1.$$

Since $y \in \mathrm{cl}(X_1 \cap Y_2)$ we can easily deduce that $y \in \mathrm{cl}((X_1 \cap Y_1) \cup x)$, contrary to our initial assumption. □

Lemma 4.8. *Suppose that C^* is a cocircuit of the 3-connected matroid M. Suppose that (X_1, X_2, x) is a minimal partition of M with respect to C^*. Assume that $\mathrm{si}(M/x_0)$ is not 3-connected for any element $x_0 \in C^* \cap X_1$. Let (Y_1, Y_2, y) be a vertical 3-partition of M such that $y \in C^* \cap X_1$, and assume that $x \in Y_1$. Then $|X_1 \cap Y_2| = 1$.*

Proof. The hypotheses of the lemma imply that every circuit and cocircuit of M contains at least three elements. Let us assume that the lemma fails, so that $|X_1 \cap Y_2| \geq 2$. Now (v) of Lemma 4.5 implies that $(X_1 \cap Y_2) \cup y$ contains a triangle of M that contains y. Since C^* meets this triangle in y, there must be an element $z \in X_1 \cap Y_2$ such that $z \in C^*$.

By assumption $\mathrm{si}(M/z)$ is not 3-connected so Proposition 2.5 implies that there is vertical 3-partition (Z_1', Z_2', z). Let us assume that $x \in Z_1'$.

Suppose that $y \in Z_i'$, where $\{i, j\} = \{1, 2\}$. Since $\mathrm{r}((X_1 \cap Y_2) \cup y) = 2$ and $z \in \mathrm{cl}(Z_i')$ it follows that $(X_1 \cap Y_2) \cup y \subseteq \mathrm{cl}(Z_i')$, as $y \neq z$ and $z \in X_1 \cap Y_2$. Let $Z_i = Z_i' \cup (X_1 \cap Y_2) \cup y$ and let $Z_j = Z_j' - Z_i$. Then Proposition 2.6 implies that (Z_1, Z_2, z) is a vertical 3-partition. Note that $x \in Z_1$, whether i is equal to 1 or 2.

Suppose that $i = 2$. Then $(X_1 \cap Y_2) \cup y \subseteq Z_2 \cup z$. This means that $(X_1 \cap Z_1) \cup x \subseteq (X_1 \cap Y_1) \cup \{x, y\}$. Lemma 4.7 says that $z \in \mathrm{cl}((X_1 \cap Z_1) \cup x)$. Therefore $z \in \mathrm{cl}((X_1 \cap Y_1) \cup \{x, y\})$. But since $\{y, z\}$ spans $(X_1 \cap Y_2) \cup y$ this implies that $(X_1 \cap Y_1) \cup \{x, y\}$ spans $X_1 \cap Y_2$. As $x, y \in \mathrm{cl}(Y_1)$ it now follows that Y_1 spans $X_1 \cap Y_2$, in contradiction to Lemma 4.5 (iv). Therefore $i = 1$, so $(X_1 \cap Y_2) \cup y \subseteq Z_1 \cup z$.

We conclude that $X_1 \cap Z_2 \subseteq (X_1 \cap Y_1) \cup \{x, y\}$. Suppose that $|X_1 \cap Z_2| \geq 2$. It follows from (v) of Lemma 4.5 that $\mathrm{r}((X_1 \cap Z_2) \cup z) = 2$. Therefore z is in $\mathrm{cl}(X_1 \cap Z_2)$, and hence in $\mathrm{cl}((X_1 \cap Y_1) \cup \{x, y\})$. Exactly as before, we conclude that Y_1 spans $X_1 \cap Y_2$, a contradiction. Therefore $|X_1 \cap Z_2| \leq 1$.

As $\mathrm{r}(Z_2) \geq 3$ we deduce that $|X_2 \cap Z_2| \geq 2$. But $\lambda(X_2 \cap Z_2) \leq 2$ by (ii) of Lemma 4.5, so it follows that $\lambda(X_2 \cap Z_2) = 2$, and hence $\lambda(X_1 \cup Z_1) = 2$. Now $\lambda(X_1 \cup x) + \lambda(Z_1 \cup z) = 4$, so the submodularity of the connectivity function implies that

$$\lambda((X_1 \cap Z_1) \cup \{x, z\}) + \lambda(X_1 \cup Z_1) \leq 4.$$

We now conclude that $\lambda((X_1 \cap Z_1) \cup \{x, z\}) \leq 2$. It follows from (vi) of Lemma 4.5 that $\mathrm{r}((X_1 \cap Z_1) \cup \{x, z\}) = 2$.

We have already deduced that $(X_1 \cap Y_2) \cup y \subseteq Z_1 \cup z$, so $X_1 \cap Y_2 \subseteq (X_1 \cap Z_1) \cup z$. But $|X_1 \cap Y_2| \geq 2$, and $\mathrm{r}((X_1 \cap Z_1) \cup \{x, z\}) = 2$. Therefore $x \in \mathrm{cl}(X_1 \cap Y_2)$. We also know that $y \in \mathrm{cl}(X_1 \cap Y_2)$. Proposition 4.6 asserts that

$$\sqcap((X_1 \cap Y_1) \cup \{x, y\}, X_1 \cap Y_2) = 1.$$

Since $x, y \in \mathrm{cl}(X_1 \cap Y_2)$ it follows from Proposition 2.14 that $\mathrm{r}(\{x, y\}) \leq 1$, a contradiction as M is 3-connected. This completes the proof of the lemma. □

5. Proof of the main result

We restate Theorem 1.2 here.

Theorem 5.1. *Suppose that M and N are 3-connected matroids such that $|E(N)| \geq 4$ and C^* is a cocircuit of M with the property that M/x_0 has an N-minor for some $x_0 \in C^*$. Then either:*

(i) there is an element $x \in C^*$ such that $\mathrm{si}(M/x)$ is 3-connected and has an N-minor;

(ii) *there is a four-element fan* (x_1, x_2, x_3, x_4) *of* M *such that* $x_1, x_3 \in C^*$*, and* $\mathrm{si}(M/x_2)$ *is* 3*-connected with an* N*-minor;*

(iii) *there is a segment-cosegment pair* (L, L^*) *such that* $L \subseteq C^*$*, and* $\mathrm{cl}(L) - L$ *contains a single element* e*. In this case* $e \notin C^*$ *and* $\mathrm{si}(M/e)$ *is* 3*-connected with an* N*-minor. Moreover* $M/\mathrm{cl}(L)$ *is* 3*-connected with an* N*-minor, and if* $x_i \in L$ *then* M/x_i *is* 3*-connected up to a unique spore* $(\mathrm{cl}(L) - x_i, y_i)$*; or,*

(iv) *there is a segment-cosegment pair* (L, L^*) *such that* L *is a flat and* $|L - C^*| \leq 1$*. In this case* M/L *is* 3*-connected with an* N*-minor, and if* $x_i \in L$ *then* M/x_i *is* 3*-connected up to a unique spore* $(L - x_i, y_i)$*.*

Proof. Assume that M is a counterexample to the theorem. Let x_0 be an element of C^* such that N is a minor of M/x_0. By hypothesis $\mathrm{si}(M/x_0)$ is not 3-connected, so Proposition 2.5 implies there is a vertical 3-partition (Z_1, Z_2, x_0). It follows easily that $|E(M)| \geq 7$. By Proposition 2.9 we will assume, relabeling as necessary, that $|E(N) \cap Z_1| \leq 1$. Let $Z = Z_1 - \mathrm{cl}(Z_2)$. Lemma 2.10 implies that M/e has an N-minor for every element $e \in Z$, and Lemma 4.3 implies that there is a minimal partition (X_1, X_2, x) with respect to C^* such that $x \in C^* \cap (Z \cup x_0)$, and $X_1 \subseteq Z$.

Proposition 4.1 implies that C^* has a non-empty intersection with $X_1 - \mathrm{cl}(X_2)$. If $s \in C^* \cap (X_1 - \mathrm{cl}(X_2))$ then $\mathrm{si}(M/s)$ is not 3-connected by hypothesis. Therefore there is a vertical 3-partition (S_1, S_2, s).

5.1.1. *Suppose that* $s \in C^*$ *is contained in* $X_1 - \mathrm{cl}(X_2)$ *and that* (S_1, S_2, s) *is a vertical* 3*-partition such that* $x \in S_1$*. Then* $|X_1 \cap S_1| \geq 2$ *and* $(X_1 \cap S_1) \cup \{s, x\}$ *is a segment of* M*.*

Proof. Lemma 4.8 tells us that $|X_1 \cap S_2| = 1$. By Lemma 4.5 (i) we know that $|X_1 \cap S_1| \geq 1$. Assume that $|X_1 \cap S_1| = 1$. Then X_1 contains exactly three elements: the unique element in $X_1 \cap S_2$, the unique element in $X_1 \cap S_1$, and s. By the definition of a vertical 3-partition it follows that $\mathrm{r}(X_1) = 3$ and that X_1 is a triad of M. As $x \in \mathrm{cl}(X_1)$ it follows that there is a circuit $C \subseteq X_1 \cup x$ that contains x. It cannot be the case that the single element in $X_1 \cap S_2$ is in C, for that would imply that $X_1 \cap S_2 \subseteq \mathrm{cl}(S_1)$, contradicting Lemma 4.5 (iv). As C does not meet the triad X_1 in a single element it follows that $(X_1 \cap S_1) \cup \{x, s\}$ is a triangle.

If we let x_2 be the unique element in $X_1 \cap S_1$, let x_4 be the unique element in $X_1 \cap S_2$, and let $x_1 = x$ and $x_3 = s$, then (x_1, x_2, x_3, x_4) is a four-element fan of M. If $\mathrm{si}(M/x_2)$ is 3-connected then statement (ii)

of Theorem 5.1 holds, which is a contradiction as M is a counterexample to the theorem. Therefore we will assume that $\mathrm{si}(M/x_2)$ is not 3-connected.

Since $\mathrm{si}(M/x_3)$ is not 3-connected Theorem 2.15 asserts that $\mathrm{co}(M\backslash x_3)$ is 3-connected. Assume that every triad of M that contains x_3 also contains x_2. Then $\mathrm{co}(M\backslash x_3) \cong M\backslash x_3/x_2$. However x_3 is contained in a parallel pair in M/x_2, so $\mathrm{si}(M/x_2)$ is obtained from $M\backslash x_3/x_2$ by possibly deleting parallel elements. As $M\backslash x_3/x_2$ is 3-connected it follows that $\mathrm{si}(M/x_2)$ is 3-connected, contrary to hypothesis.

Therefore there is a triad T^* of M that contains x_3 but not x_2. Now T^* cannot meet the triangle $\{x_1, x_2, x_3\}$ in exactly one element, and therefore $x_1 \in T^*$. Let y_2 be the unique element in $T^* - \{x_1, x_3\}$. Since every triad that contains x_3 must contain either x_1 or x_2, and since both $\{x_1, x_3\}$ and $\{x_2, x_3\}$ are contained in triads of M it follows that $\mathrm{co}(M\backslash x_3) \cong M\backslash x_3/x_1/x_2$. Note that x_3 is a loop of $M/x_1/x_2$, so $M\backslash x_3/x_1/x_2 = M/x_3/x_1/x_2$.

As $\mathrm{si}(M/x_3)$ is not 3-connected there is a vertical 3-partition (Z_1, Z_2, x_3) of M. By relabeling as necessary we may assume that $x_1 \in Z_2$. Hence $x_2 \in \mathrm{cl}(Z_2 \cup x_3)$, so by Proposition 2.6 we may assume that $x_2 \in Z_2$. Now (Z_1, Z_2) is an exact 2-separation of M/x_3, but $M/x_3/x_1/x_2$ is 3-connected. By Proposition 2.1 we see that $Z_2 - \{x_1, x_2\}$ must contain at most one element. If $Z_2 = \{x_1, x_2\}$ then $\mathrm{r}(Z_2) \leq 2$, a contradiction. Therefore $Z_2 - \{x_1, x_2\}$ contains exactly one element. Let this element be y_3. It is easy to see that Z_2 must be a triad of M.

We relabel x_4 with y_1. Let $L = \{x_1, x_2, x_3\}$ and let $L^* = \{y_1, y_2, y_3\}$. Now L is a segment of M. Proposition 4.4 implies $X_2 \cup x_1$ is a hyperplane, and as $\{x_1, x_2, x_3\}$ is a triangle it is easy to see that $\sqcap(X_2 \cup x_1, \{x_2, x_3\}) = 1$. If there were some element e in $\mathrm{cl}(L) - L$ then Proposition 2.14 would imply that $\mathrm{r}(\{e, x_1\}) \leq 1$, a contradiction. Therefore L is a flat of M. Moreover $(L - x_i) \cup y_i$ is a cocircuit of M for all $i \in \{1, 2, 3\}$, so (L, L^*) is a segment-cosegment pair of M.

By applying Proposition 3.3 and Lemma 3.5 we see that M/L is 3-connected, and that M/x_i is 3-connected up to a unique spore $(L - x_i, y_i)$ for all $i \in \{1, 2, 3\}$. We know that M/x_3 has an N-minor. However $\{x_1, x_2\}$ is a parallel pair in M/x_3, so $M/x_3\backslash x_1$ has an N-minor. Furthermore $\{x_2, y_3\}$ is a series pair of $M/x_3\backslash x_1$, so $M/x_3\backslash x_1/x_2$, and hence M/L, has an N-minor. Thus statement (iv) of Theorem 5.1 holds, a contradiction. We conclude that $|X_1 \cap S_1| \geq 2$.

Since $\lambda(X_1 \cup x) = \lambda(S_1 \cup s) = 2$ it follows that

$$\lambda((X_1 \cap S_1) \cup \{s, x\}) + \lambda(X_1 \cup S_1) \leq 4.$$

Suppose that $\lambda((X_1 \cap S_1) \cup \{s, x\}) \geq 3$. Then $\lambda(X_1 \cup S_1) \leq 1$, so $\lambda(X_2 \cap S_2) \leq 1$. However, as $|X_1 \cap S_2| = 1$ it follows that $|X_2 \cap S_2| \geq 2$, so M contains a 2-separation, a contradiction. Thus $\lambda((X_1 \cap S_1) \cup \{s, x\}) \leq 2$ and it follows from Lemma 4.5 (vi) that $(X_1 \cap S_1) \cup \{s, x\}$ is a segment. □

5.1.2. *The rank of $X_1 \cup x$ is three. Moreover, X_1 is a cocircuit of M.*

Proof. Let $s \in C^*$ be an element in $X_1 - \mathrm{cl}(X_2)$ and suppose that (S_1, S_2, s) is a vertical 3-partition such that $x \in S_1$. Then $\mathrm{r}((X_1 \cap S_1) \cup \{s, x\}) = 2$ by 5.1.1, and as $|X_1 \cap S_2| = 1$, Lemma 4.5 (iv) implies that $\mathrm{r}(X_1 \cup x) = 3$.

Proposition 4.4 asserts that $X_2 \cup x$ is a flat of M, so X_1 is a cocircuit. □

5.1.3. *Suppose that y and z are elements in $C^* \cap X_1$, and (Y_1, Y_2, y) and (Z_1, Z_2, z) are vertical 3-partitions such that $x \in Y_1 \cap Z_1$. Then*

$$|X_1 \cap Y_2| = |X_1 \cap Z_2| = 1 \quad \textit{and} \quad X_1 \cap Y_2 = X_1 \cap Z_2.$$

Moreover

$$(X_1 \cap Y_1) \cup \{x, y\} = (X_1 \cap Z_1) \cup \{x, z\}.$$

Proof. Let x' be the unique element in $X_1 \cap Y_2$. From 5.1.1 we see that $(X_1 \cap Y_1) \cup \{x, y\}$ is a segment. The only element of X_1 not in $(X_1 \cap Y_1) \cup \{x, y\}$ is x'. It cannot be the case that $x' \in \mathrm{cl}((X_1 \cap Y_1) \cup \{x, y\})$ by Lemma 4.5 (vi). The same arguments shows that $(X_1 \cap Z_1) \cup \{x, z\}$ is a segment, and the only element of X_1 not in this segment is x'. Now the result follows easily. □

5.1.4. *Let $y \in C^*$ be an element in X_1 and suppose that (Y_1, Y_2, y) is a vertical 3-partition such that $x \in Y_1$. Then $|X_2 \cap Y_1| = 1$.*

Proof. We know by 5.1.1 that $(X_1 \cap Y_1) \cup \{x, y\}$ is a segment. Let $L' = (X_1 \cap Y_1) \cup \{x, y\}$ and let x' be the unique element in $X_1 \cap Y_2$. Since the complement of C^* is a flat of M which does not contain the segment L' it follows that at most one element of L' is not contained in C^*. As $|X_1 \cap Y_1| \geq 2$ we can find an element $z \in (X_1 \cap Y_1) \cap C^*$. There must be a vertical 3-partition (Z_1, Z_2, z) such that $x \in Z_1$. From 5.1.3 we see that the unique element in $X_1 \cap Z_2$ is x', and that $(X_1 \cap Z_1) \cup \{x, z\} = L'$.

Let Y_i' and Z_i' denote $X_2 \cap Y_i$ and $X_2 \cap Z_i$ respectively for $i = 1, 2$. As (X_1, X_2, x) is a minimal partition it follows that Y_i' and Z_i' are non-empty for all $i \in \{1, 2\}$. Henceforth we will assume that $|Y_1'| > 1$ in order to obtain a contradiction.

5.1.5. $x \in \mathrm{cl}(Y_1')$.

Proof. We know that $\lambda(Y_1' \cup x) \leq 2$ by Lemma 4.5 (ii). Since $|Y_1'| \geq 2$ it follows that $\lambda(Y_1' \cup x) = 2$ and hence $\lambda(X_1 \cup Y_2) = 2$. Since $x \in \mathrm{cl}(X_1 \cup Y_2)$ it follows that $\lambda(Y_1') = 2$, so Lemma 2.2 implies that $x \in \mathrm{cl}(Y_1')$. □

5.1.6. *Neither $Y_1' \cap Z_1'$ nor $Y_2' \cap Z_2'$ is empty.*

Proof. We know from 5.1.5 that $x \in \mathrm{cl}(Y_1')$. Since $z \in \mathrm{cl}(Z_2)$ but $(X_1 \cap Z_1) \nsubseteq \mathrm{cl}(Z_2)$, we deduce that $x \notin \mathrm{cl}(Z_2)$ as L' is a segment containing both x and z. Thus $x \notin \mathrm{cl}(Z_2' \cup x')$. Hence $Y_1' - Z_2' \neq \emptyset$ so $Y_1' \cap Z_1' \neq \emptyset$.

Note that z is in the closure of $Z_2 = Z_2' \cup x'$, but $z \notin \mathrm{cl}(Z_2')$ as X_1 is a cocircuit by 5.1.2. This observation means that $x' \in \mathrm{cl}(Z_2' \cup z)$. However $z \in Y_1$, and $x' \notin \mathrm{cl}(Y_1)$ by Lemma 4.5 (iv). Thus $x' \notin \mathrm{cl}(Y_1' \cup z)$. It follows that $Z_2' - Y_1' \neq \emptyset$, so $Z_2' \cap Y_2' \neq \emptyset$. □

5.1.7. *$(L' \cup (Y_1' \cap Z_1'), Y_2 \cup Z_2)$ is a 3-separation of M.*

Proof. Note that $\lambda(Y_2) = \lambda(Z_2) = 2$, so $\lambda(Y_2 \cap Z_2) + \lambda(Y_2 \cup Z_2) \leq 4$. From 5.1.6 we see that $Y_2' \cap Z_2' \neq \emptyset$. Moreover $x' \in (Y_2 \cap Z_2) - (Y_2' \cap Z_2')$, which implies that $|Y_2 \cap Z_2| \geq 2$. Thus $\lambda(Y_2 \cap Z_2) \geq 2$, so $\lambda(Y_2 \cup Z_2) \leq 2$. As both $L' \cup (Y_1' \cap Z_1')$ and $Y_2 \cup Z_2$ have cardinality at least three the claim follows. □

Note that $y, z \in \mathrm{cl}(Y_2 \cup Z_2)$. As y and z are contained in the segment L' it follows that $L' \subseteq \mathrm{cl}(Y_2 \cup Z_2)$. If $|Y_1' \cap Z_1'| \geq 2$ then it must be the case that $L' \subseteq \mathrm{cl}(Y_1' \cap Z_1')$, for otherwise $(Y_1' \cap Z_1', (Y_2 \cup Z_2) \cup L')$ is a 2-separation of M. But $L' \subseteq \mathrm{cl}(Y_1' \cap Z_1')$ implies that $X_1 \cap Y_1 \subseteq \mathrm{cl}(X_2)$, a contradiction.

Therefore $|Y_1' \cap Z_1'| \leq 1$. We know from 5.1.6 that $Y_1' \cap Z_1'$ is not empty. Let e be the unique element in $Y_1' \cap Z_1'$. Suppose that $e \in \mathrm{cl}(L')$. As $X_2 \cup x$ is a hyperplane and L' is a segment we see that $\sqcap(X_2 \cup x, L' - x) = 1$. As $e, x \in \mathrm{cl}(L' - x)$ it follows from Proposition 2.14 that $\mathrm{r}(\{e, x\}) \leq 1$. We deduce from this contradiction that $e \notin \mathrm{cl}(L')$.

Hence $\mathrm{r}(L' \cup e) = 3$, so $\mathrm{r}(Y_2 \cup Z_2) = \mathrm{r}(M) - 1$ by 5.1.7. Thus the complement of $\mathrm{cl}(Y_2 \cup Z_2)$ is a cocircuit. However $L' \subseteq \mathrm{cl}(Y_2 \cup Z_2)$, so e is a coloop of M, a contradiction.

Our assumption that $|X_2 \cap Y_1| \geq 2$ has lead to an impossibility. Since $X_2 \cap Y_1$ is non-empty by Lemma 4.5 (i) we conclude that 5.1.4 is true. □

Now we are in a position to complete the proof of Theorem 5.1. Let $x_1 = x$, and let x_2 be some element in $C^* \cap X_1$. There is a vertical 3-partition (Y_1^2, Y_2^2, x_2) such that $x_1 \in Y_1^2$. Lemma 4.8 tells us that $|X_1 \cap Y_2^2| = 1$. Let y_1 be the unique element in $X_1 \cap Y_2^2$.

We know that $|X_1 \cap Y_1^2| \geq 2$ and $(X_1 \cap Y_1^2) \cup \{x_1, x_2\}$ is a segment by 5.1.1. It follows from Proposition 2.14, and the fact that $(X_1 \cap Y_1^2) \cup x_2$ is a segment while $X_2 \cup x_1$ is a hyperplane, that $(X_1 \cap Y_1^2) \cup \{x_1, x_2\}$ is a flat. The complement of C^* can contain at most one element of $(X_1 \cap Y_1^2) \cup \{x_1, x_2\}$. Let $L = C^* \cap ((X_1 \cap Y_1^2) \cup \{x_1, x_2\})$. Then $\mathrm{cl}(L) = (X_1 \cap Y_1^2) \cup \{x_1, x_2\}$, and $\mathrm{cl}(L) - L$ contains at most one element.

Suppose that $L = \{x_1, \ldots, x_t\}$. We know that $t \geq 3$. Let i be a member of $\{2, \ldots, t\}$. As $x_i \in C^*$ the fact that M is a counterexample to the theorem means that $\mathrm{si}(M/x_i)$ is not 3-connected, so there is a vertical 3-partition (Y_1^i, Y_2^i, x_i) such that $x_1 \in Y_1^i$. Then

$$(X_1 \cap Y_1^i) \cup \{x_1, x_i\} = (X_1 \cap Y_1^2) \cup \{x_1, x_2\}$$

by 5.1.3, and 5.1.4 implies that there is a unique element in $X_2 \cap Y_1^i$. Let y_i be this element.

Define L^* to be $\{y_1, \ldots, y_t\}$. Note that $L \cap L^* = \emptyset$. We already know that $(\mathrm{cl}(L) - x_1) \cup y_1 = X_1$ is a cocircuit. Suppose that $i \in \{2, \ldots, t\}$. Then $(\mathrm{cl}(L) - x_i) \cup y_i$ is Y_1^i. As Y_1^i contains only one element that is not in the segment $\mathrm{cl}(L)$ it follows that $\mathrm{r}(Y_1^i) = 3$. Thus $\mathrm{r}(Y_2^i \cup x_i) = r(M) - 1$. Furthermore $Y_2^i \cup x_i$ is a flat, for otherwise the complement of $\mathrm{cl}(Y_2^i \cup x_i)$ is a cocircuit of rank at most two, which cannot occur since M is 3-connected. Hence $(\mathrm{cl}(L) - x_i) \cup y_i$ is a cocircuit.

We have shown that (L, L^*) is a segment-cosegment pair. Proposition 3.3 says that $M/\mathrm{cl}(L)$ is 3-connected. It is easy to see that the hypotheses of Lemma 3.5 are satisfied, so M/x_i is 3-connected up to the unique spore $(\mathrm{cl}(L) - x_i, y_i)$, for all $i \in \{1, \ldots, t\}$. We know that M/x_2 has an N-minor, but as $\mathrm{cl}(L) - x_2$ is a parallel class of M/x_2 it follows that $M/x_2 \backslash (\mathrm{cl}(L) - \{x_1, x_2\})$ has an N-minor. Since $\{x_1, y_2\}$ is a series pair of $M/x_2 \backslash (\mathrm{cl}(L) - \{x_1, x_2\})$ it follows that $M/x_2 \backslash (\mathrm{cl}(L) - \{x_1, x_2\})/x_1$, and hence $M/\mathrm{cl}(L)$, has an N-minor.

Suppose that $|\mathrm{cl}(L) - C^*| = 0$. Then $L = \mathrm{cl}(L)$, and statement (iv) of Theorem 5.1 holds. Therefore we must assume that there is a single element e in $\mathrm{cl}(L) - L$. Lemma 2.10 tells us that M/e has an N-minor. If $\mathrm{si}(M/e)$ is 3-connected, then statement (iii) holds. Therefore we must assume $\mathrm{si}(M/e)$ is not 3-connected.

Let $x_{t+1} = e$. There must be a vertical 3-partition $(Y_1^{t+1}, Y_2^{t+1}, x_{t+1})$. We assume that $x_1 \in Y_1^{t+1}$. Since $\mathrm{cl}(Y_1^{t+1})$ contains x_1 and x_{t+1} it follows that $\mathrm{cl}(L) \subseteq \mathrm{cl}(Y_1^{t+1})$. By Proposition 2.6 we may assume that Y_1^{t+1} contains $\mathrm{cl}(L) - x_{t+1} = L$.

As $X_2 \cup x_1$ is a flat it follows that $x_{t+1} \notin \mathrm{cl}(X_2)$. However $x_{t+1} \in \mathrm{cl}(Y_2^{t+1})$, so $X_1 \cap Y_2^{t+1} \neq \emptyset$. We know that $X_1 = (L \cup \{x_{t+1}, y_1\}) - x_1$, and as $L \subseteq Y_1^{t+1}$ it follows that $X_1 \cap Y_2^{t+1} = \{y_1\}$.

Since $x_{t+1} \in \mathrm{cl}(Y_2^{t+1})$, there is a circuit $C_1 \subseteq Y_2^{t+1} \cup x_{t+1}$ such that $x_{t+1} \in C_1$. But $Y_1^2 = (L \cup \{x_{t+1}, y_2\}) - x_2$ is a cocircuit of M and C_1 must meet this cocircuit in more than one element. The only element of $Y_1^2 - x_{t+1}$ that can be in C_1 is y_2. Thus $y_2 \in Y_2^{t+1}$.

Since (X_1, X_2, x) is a minimal partition it follows that $X_2 \cap Y_1^{t+1}$ is non-empty. Assume that $|X_2 \cap Y_1^{t+1}| \geq 2$. As $\lambda(X_1) + \lambda(Y_2^{t+1} \cup x_{t+1}) = 4$, it follows that

$$\lambda((X_1 \cap Y_2^{t+1}) \cup x_{t+1}) + \lambda(X_1 \cup Y_2^{t+1}) \leq 4.$$

Furthermore $\lambda(X_1 \cup x_1) + \lambda(Y_2^{t+1} \cup x_{t+1}) = 4$, so

$$\lambda((X_1 \cap Y_2^{t+1}) \cup x_{t+1}) + \lambda(X_1 \cup Y_2^{t+1} \cup x_1) \leq 4.$$

As $(X_1 \cap Y_2^{t+1}) \cup x_{t+1} = \{x_{t+1}, y_1\}$ we deduce that $\lambda((X_1 \cap Y_2^{t+1}) \cup x_{t+1}) = 2$. Thus

$$\lambda(X_1 \cup Y_2^{t+1}),\ \lambda(X_1 \cup Y_2^{t+1} \cup x_1) \leq 2. \tag{1}$$

Both of the sets in Equation (1) contain at least two elements, and by assumption $|X_2 \cap Y_1^{t+1}| \geq 2$. Therefore $X_2 \cap Y_1^{t+1}$ and $(X_2 \cap Y_1^{t+1}) \cup x_1$ are exactly 3-separating. Since $x_1 \in \mathrm{cl}(X_1)$ we see from Lemma 2.2 that $x_1 \in \mathrm{cl}(X_2 \cap Y_1^{t+1})$. Thus there is a circuit $C_2 \subseteq (X_2 \cap Y_1^{t+1}) \cup x_1$ such that $x_1 \subseteq C_2$. We have already noted that Y_1^2 is a cocircuit, and as $x_1 \in Y_1^2$ it follows that $|C_2 \cap Y_1^2| \geq 2$. As $C_2 - x_1 \subseteq X_2$ the only element other than x_1 that can be in $C_2 \cap Y_1^2$ is y_2. Hence $y_2 \in C_2 \subseteq Y_1^{t+1}$, a contradiction as we have already deduced that $y_2 \in Y_2^{t+1}$.

We are forced to conclude that $X_2 \cap Y_1^{t+1}$ contains a unique element. Let this element be y_{t+1}. Therefore $Y_1^{t+1} = L \cup y_{t+1}$. Thus $\mathrm{r}(Y_1^{t+1}) = 3$, so $\mathrm{r}(Y_2^{t+1}) = \mathrm{r}(M) - 1$. If $Y_2^{t+1} \cup x_{t+1}$ is not a hyperplane, then the complement of $\mathrm{cl}(Y_2^{t+1} \cup x_{t+1})$ is a cocircuit of rank at most two, a contradiction. Therefore $(\mathrm{cl}(L) - x_{t+1}) \cup y_{t+1} = Y_1^{t+1}$ is a cocircuit.

Let $L_0 = \{x_1, \ldots, x_{t+1}\}$ and let $L_0^* = \{y_1, \ldots, y_{t+1}\}$. Note that $L_0 = \mathrm{cl}(L)$, so L_0 is a flat. We have shown that (L_0, L_0^*) is a segment-cosegment pair. Moreover, M/x_{t+1} is 3-connected up to a unique spore $(L_0 - x_{t+1}, y_{t+1})$, by Lemma 3.5. By relabeling L_0 and L_0^* as L and L^* respectively we see that statement (iv) of Theorem 5.1 holds. Hence M is not a counterexample, and this contradiction completes the proof of Theorem 5.1. □

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